# A note on the application of the supersonic area rule to the determination of the wave drag of rectangular wings

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(*Received* 12 *April* 1957)

SUMMARY

A proof is given that, in spite of certain singularities in the oblique area distributions, the supersonic area rule can be used to determine correctly the wave drag, according to linearized theory, of thin symmetrical rectangular wings at zero incidence. The proof could be extended to cover the case of swept wings with supersonic edges, for which similar singularities also exist.

## 1. INTRODUCTION

It has recently been suggested that the 'supersonic area rule' (Jones 1953), when used to determine the wave drag at zero lift of wings with straight supersonic edges, gives results which disagree with those of conventional thin wing theory, due to certain singularities which appear in the oblique area distributions. It is the purpose of the present note to show that this is not the case, provided that these singularities are handled with care. A proof is given of the equivalence of the wave drag as calculated by the two methods for unswept wings of rectangular planform; the proof could easily be extended to cover the case of untapered swept wings with supersonic leading and trailing edges.

## 2. THE SUPERSONIC AREA RULE

We consider a rectangular wing of unit chord and span $b = 2s$, and use standard rectangular axes with origin $O$ at the centre of the leading edge, $Ox$ in the direction of the free stream, $Oy$ along the leading edge and $Oz$ vertically upward; we shall also use cylindrical coordinates $r = \sqrt{(y^2+z^2)}$ and $\theta = \tan^{-1}z/y$. The wing section, which must be symmetrical with sharp leading and trailing edges, is taken to be

$$z = \pm f(x) \qquad (0 \leqslant x \leqslant 1), \tag{1}$$

and we define $f(x)$ to be zero outside this range.

The area rule for calculating the wave drag, according to the linearized theory of supersonic flow, of any body which can be represented solely by a distribution of sources, may be deduced from considerations of momentum flux across the large cylinder $r = R$; use is made of Hayes's theorem on equivalent source positions (see, for example, Heaslet, Lomax & Spreiter 1952), which states that, for closed source distributions at large values of $r$ and for a given value of $\theta$, the flow due to a unit source is independent of its position on the plane

$$x = x_0 + \beta(y\cos\theta + z\sin\theta), \tag{2}$$

where $x_0$ is constant and $\beta = \sqrt{(M^2-1)}$. When applied to plane wings, for which the source strength is proportional to the local slope of the surface in the $x$-direction, the theorem shows that the planar source distribution may be concentrated onto the $x$-axis so that, again for a given value of $\theta$, the effect at large distances is the same as that of an 'equivalent body of revolution' whose cross-sectional area at $x = \xi$ is $S(\xi,\theta)$, defined as the area of the projection on a plane normal to the axis $Ox$ of the section of the wing by the plane.

$$x = \xi + \beta y\cos\theta. \tag{3}$$

In this way the following well-known formula for the wave drag $D$ may be obtained:

$$\frac{D}{\frac{1}{2}\rho_0 U^2} = -\frac{1}{4\pi^2}\int_0^{2\pi} d\theta \iint d\xi d\eta \log|\xi-\eta| S''(\xi,\theta)S''(\eta,\theta), \tag{4}$$

where $\rho_0$ and $U$ are the density and velocity of the free stream, $S(\xi,\theta)$ is defined above, and primes denote differentiation with respect to $\xi$ (or $\eta$). For symmetrical unyawed wings this reduces to

$$\frac{D}{\frac{1}{2}\rho_0 U^2} = -\frac{1}{\pi^2}\int_0^{\frac{1}{2}\pi} d\theta \iint d\xi d\eta \log|\xi-\eta| S''(\xi,\theta)S''(\eta,\theta). \tag{5}$$

In order that equations (4) and (5) shall be valid, it is necessary that $S'(\xi,\theta)$ shall be a continuous function of $\xi$, though $S''(\xi,\theta)$ may have a finite number of discontinuities. But it is evident that, in the case of a rectangular wing, this requirement is not satisfied when $\theta = \frac{1}{2}\pi$; $S'(\xi,\frac{1}{2}\pi)$ is in fact proportional to $f'(\xi)$, with discontinuities at $\xi = 0$ and 1. To put this in another way, we see that when the source distribution representing the wing is concentrated onto the axis along lines $x = \xi$ (cf. (3) with $\theta = \frac{1}{2}\pi$) *parallel to the leading edge*, the resulting axial distribution is of finite strength at the leading and trailing edges; and such a distribution is inadmissible since it leads to infinite velocities on the Mach cones from these points. Physically, the situation is due to the fact that (according to unmodified linearized theory), the flow in the plane $\theta = \frac{1}{2}\pi$ is always partly two-dimensional in character, with leading and trailing edge shocks of first order strength, however large $R$ may be; and such a flow can never be represented by a continuous distribution of purely axial sources. The effect is however concentrated, so far as the control surface $r = R$ is concerned, in the immediate neighbourhood of the points $(\beta R, 0, \pm R)$ and $(\beta R + 1, 0, \pm R)$ (when $R$ is sufficiently large); and such isolated singularities do not invalidate the use of equations (4) and (5) for the overall wave drag, provided that care is taken near $\theta = \pm\frac{1}{2}\pi$.

## 3. Determination of oblique area distributions

It is convenient to write

$$\sigma(\theta) = \beta s \cos\theta.$$

There are two principal cases to be considered, according as $\sigma \gtrless \frac{1}{2}$.

(*a*) $\sigma \geqslant \frac{1}{2}$ or $|\sec\theta| \leqslant 2\beta s$ (see figure 1 *a*). Since the sections in question are simply oblique sections of a cylinder, it is easily seen that

$$S(\xi,\theta) = \frac{2\sec\theta}{\beta}\int_0^{\xi+\sigma} f(x)\,dx \qquad (-\sigma \leqslant \xi \leqslant 1-\sigma) \tag{6a}$$

$$= \frac{2\sec\theta}{\beta}\int_0^{1} f(x)\,dx \qquad (1-\sigma \leqslant \xi \leqslant \sigma) \tag{7a}$$

$$= \frac{2\sec\theta}{\beta}\int_{\xi-\sigma}^{1} f(x)\,dx \qquad (\sigma \leqslant \xi \leqslant 1+\sigma). \tag{8a}$$

(*b*) $0 < \sigma \leqslant \frac{1}{2}$ or $|\sec\theta| \geqslant 2\beta s$ (see figure 1 *b*). In this case the sections for $\sigma \leqslant \xi \leqslant 1-\sigma$ cut both wing tips, and the results have to be modified as follows:

$$S(\xi,\theta) = \frac{2\sec\theta}{\beta}\int_0^{\xi+\sigma} f(x)\,dx \qquad (-\sigma \leqslant \xi \leqslant \sigma) \tag{6b}$$

$$= \frac{2\sec\theta}{\beta}\int_{\xi-\sigma}^{\xi+\sigma} f(x)\,dx \qquad (\sigma \leqslant \xi \leqslant 1-\sigma) \tag{7b}$$

$$= \frac{2\sec\theta}{\beta}\int_{\xi-\sigma}^{1} f(x)\,dx \qquad (1-\sigma \leqslant \xi \leqslant 1+\sigma). \tag{8b}$$

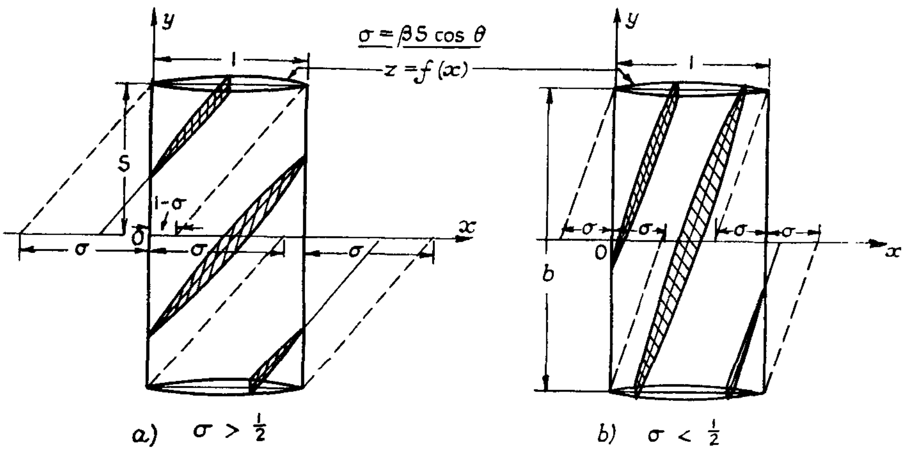


Figure 1. Determination of oblique area distributions. $S$ is the projection of the shaded area on a plane perpendicular to $Ox$.

Differentiating (6) to (8) with respect to $\xi$, we find that in both cases

$$\left.\begin{aligned} S'(\xi,\theta) &= 2\beta^{-1}\sec\theta\{f(\xi+\sigma)-f(\xi-\sigma)\} \qquad (9)\\ S''(\xi,\theta) &= 2\beta^{-1}\sec\theta\{f'(\xi+\sigma)-f'(\xi-\sigma)\} \qquad (10)\end{aligned}\right\}\ 0 \leqslant \theta < \tfrac{1}{2}\pi,$$

remembering that $f(x) = f'(x) = 0$ for $x < 0$ and $x > 1$. When $\theta = \frac{1}{2}\pi$, it is evident that

$$S'(\xi, \tfrac{1}{2}\pi) = 2sf'(\xi). \tag{11}$$

The function $S'(\xi, \theta)$ is sketched for some typical cases of a biconvex section in figure 2. It is clear that, except when $\theta = \frac{1}{2}\pi$, $S'(\xi, \theta)$ is everywhere continuous as required in (5), though $S''(\xi, \theta)$ has four discontinuities.

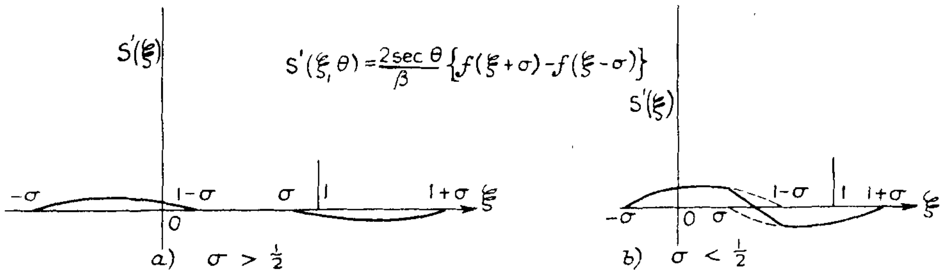


Figure 2. The function $S'(\xi, \theta)$.

## 4. Evaluation of the wave drag

Substituting from (10) in (5), we obtain

$$\frac{D}{\frac{1}{2}\rho_0 U^2} = -\frac{4}{\pi^2\beta^2}\int_0^{\frac{1}{2}\pi} \sec^2\theta\, d\theta \iint d\xi d\eta \log|\xi-\eta|\{f'(\xi+\sigma)f'(\eta+\sigma)+$$
$$+f'(\xi-\sigma)f'(\eta-\sigma)-f'(\xi+\sigma)f'(\eta-\sigma)-f'(\xi-\sigma)f'(\eta+\sigma)\}. \tag{12}$$

Each of the four double integrals with respect to $\xi$ and $\eta$ may be simplified by linear transformations of the type

$$\xi \pm \sigma = x,$$
$$\eta \pm \sigma = y,$$

leading to the result

$$\frac{D}{\frac{1}{2}\rho_0 U^2} = \frac{4}{\pi^2\beta^2}\int^{\frac{1}{2}\pi} \sec^2\theta\, d\theta \int_0^1\int_0^1 dx dy f'(x)f'(y)\log\left|1-\frac{\beta^2b^2\cos^2\theta}{(x-y)^2}\right|. \tag{13}$$

When the function $f(x)$ is given, the double integral with respect to $x$ and $y$ can be evaluated directly, and it is found that the resulting function of $\theta$ is integrable over the range 0 to $\frac{1}{2}\pi$ and yields the correct result for the wave drag $D$. But in order to obtain a general result it is clearly necessary to perform the integration with respect to $\theta$ first; this must however be done with care, since the function

$$F(\theta, x, y) \equiv \sec^2\theta \log\left|1-\frac{\beta^2b^2\cos^2\theta}{(x-y)^2}\right|,$$

although regular as $\theta \to \frac{1}{2}\pi$ if $x \neq y$ and having only logarithmic singularities if $\theta \neq \frac{1}{2}\pi$, has an essential singularity when $\theta = \frac{1}{2}\pi$, $x = y$. The triple integral (13) for $D$ cannot therefore be expressed as a volume integral, and the order of integration cannot be inverted in the usual way; but this difficulty may be overcome as follows.

An alternative expression for $D$ is

$$\frac{D}{\frac{1}{2}\rho_0 U^2} = \frac{8}{\pi^2\beta^2}\int_0^{\frac{1}{2}\pi}\sec^2\theta\,d\theta\int_0^1 f'(x)\,dx\int_0^x f'(y)\log\left|1-\frac{\beta^2b^2\cos^2\theta}{(x-y)^2}\right|dy, \quad (14)$$

and this may be written in the form

$$\frac{D}{\frac{1}{2}\rho_0 U^2} = \lim_{\epsilon\to 0}\frac{8}{\pi^2\beta^2}\{I_1(\epsilon)+I_2(\epsilon)\}, \quad (15)$$

where

$$I_1(\epsilon) = \int_0^{\frac{1}{2}\pi}\sec^2\theta\,d\theta\int_0^1 f'(x)\,dx\int_0^{x-\epsilon} f'(y)\log\left|1-\frac{\beta^2b^2\cos^2\theta}{(x-y)^2}\right|dy, \quad (16)$$

and

$$I_2(\epsilon) = \int_0^{\frac{1}{2}\pi}\sec^2\theta\,d\theta\int_0^1 f'(x)\,dx\int_{x-\epsilon}^x f'(y)\log\left|1-\frac{\beta^2b^2\cos^2\theta}{(x-y)^2}\right|dy. \quad (17)$$

The singularity at $\theta = \frac{1}{2}\pi$, $x = y$, has been isolated from the region of integration in $I_1$, so that the integration with respect to $\theta$ may be performed first.

Using the result (see Appendix)

$$\left.\begin{aligned}\int_0^{\frac{1}{2}\pi} F(\theta,x,y)\,d\theta &= -\pi \qquad (\epsilon < |x-y| \leqslant \beta b),\\ &= -\pi\left\{1-\sqrt{\left[1-\frac{\beta^2b^2}{(x-y)^2}\right]}\right\}(|x-y| \geqslant \beta b),\end{aligned}\right\} \quad (18)$$

we obtain

$$I_1(\epsilon) = -\pi\int_0^1 f'(x)\,dx\int_0^{x-\epsilon} f'(y)\,dy + \\ +\pi\int_{\beta b}^1 f'(x)\,dx\int_0^{x-\beta b} f'(y)\sqrt{\left\{1-\frac{\beta^2b^2}{(x-y)^2}\right\}}\,dy;$$

the second integral occurs only if $\beta b < 1$.

Now

$$\int_0^1 f'(x)\,dx\int_0^{x-\epsilon} f'(y)\,dy = \int_0^1 f'(x)f(x-\epsilon)\,dx = O(\epsilon),$$

since

$$\int_0^1 f'(x)f(x)\,dx = 0.$$

Hence

$$I_1(\epsilon) = \pi\int_{\beta b}^1 f'(x)\,dx\int_0^{x-\beta b} f'(y)\sqrt{\left\{1-\frac{\beta^2b^2}{(x-y)^2}\right\}}\,dy + O(\epsilon). \quad (19)$$

In evaluating $I_2$ we must retain the original order of integration. Now

$$\int_{x-\epsilon}^x f'(y)\log\left|1-\frac{\beta^2b^2\cos^2\theta}{(x-y)^2}\right|dy = f'(x)\{1+O(\epsilon)\}\int_{x-\epsilon}^x \log\left|1-\frac{\beta^2b^2\cos^2\theta}{(x-y)^2}\right|dy$$

$$= f'(x)\{1+O(\epsilon)\}\left\{\epsilon\log\left|1-\frac{\beta^2b^2}{\epsilon^2}\cos^2\theta\right| + \beta b\cos\theta\log\left|\frac{1+(\beta b/\epsilon)\cos\theta}{1-(\beta b/\epsilon)\cos\theta}\right|\right\}.$$

Thus

$$I_2(\epsilon) = \{1+O(\epsilon)\}\int_0^{\frac{1}{2}\pi}\sec^2\theta\,d\theta\int_0^1\{f'(x)\}^2\times$$

$$\times\left\{\epsilon\log\left|1-\frac{\beta^2b^2}{\epsilon^2}\cos^2\theta\right|+\beta b\cos\theta\log\left|\frac{1+(\beta b/\epsilon)\cos\theta}{1-(\beta b/\epsilon)\cos\theta}\right|\right\}dx$$

$$= \{1+O(\epsilon)\}\int_0^1\{f'(x)\}^2\,dx\times$$

$$\times\int_0^{\frac{1}{2}\pi}\sec^2\theta\left\{\epsilon\log\left|1-\frac{\beta^2b^2}{\epsilon^2}\cos^2\theta\right|+\beta b\cos\theta\log\left|\frac{1+(\beta b/\epsilon)\cos\theta}{1-(\beta b/\epsilon)\cos\theta}\right|\right\}d\theta.$$

Using the results proved in the Appendix

$$\int_0^{\frac{1}{2}\pi}\sec^2\theta\log\left|1-\frac{\beta^2b^2}{\epsilon^2}\cos^2\theta\right|d\theta = -\pi \quad (\epsilon<\beta b),$$

$$\int_0^{\frac{1}{2}\pi}\sec\theta\log\left|\frac{1+(\beta b/\epsilon)\cos\theta}{1-(\beta b/\epsilon)\cos\theta}\right|d\theta = \tfrac{1}{2}\pi^2 \quad (\epsilon<\beta b),$$

we find that

$$I_2(\epsilon) = \tfrac{1}{2}\pi^2\beta b\int_0^1\{f'(x)\}^2\,dx\,\{1+O(\epsilon)\}. \tag{20}$$

Substituting from equations (19) and (20) in (15), and taking the limit as $\epsilon\to 0$, we obtain finally

$$C_D = \frac{D}{\frac{1}{2}\rho_0U^2b} = \frac{4}{\beta}\int_0^1\{f'(x)\}^2\,dx \qquad (\beta b\geqslant 1), \tag{21 a}$$

$$= \frac{4}{\beta}\int_0^1\{f'(x)\}^2\,dx + \frac{8}{\pi\beta^2b}\int_{\beta b}^1 f'(x)\,dx\int_0^{x-\beta b}f'(y)\sqrt{\left\{1-\frac{\beta^2b^2}{(x-y)^2}\right\}}\,dy \qquad (\beta b\leqslant 1). \tag{21 b}$$

Equation (21 a) is simply the well-known result that the drag coefficient of a rectangular wing is equal to the corresponding two-dimensional value, being unaffected by the presence of the wing tips, provided that the aspect ratio is greater than $\beta^{-1}$; equation (21 b) contains the additional term that has to be added when the aspect ratio is less than $\beta^{-1}$, and it has been verified that for a biconvex section this gives results in agreement with those of Harmon (1947).

## 5. Conclusions

The argument given above shows that the area rule can be applied successfully to determine the wave drag of a rectangular wing, in spite of the singular area distribution which occurs when $\theta=\frac{1}{2}\pi$, the meaning of which has been discussed in § 2. In fact the complications introduced by this singularity in the anlysis of § 4 are due to the necessity of inverting the order of integration in (13) in order to obtain a general proof; if the integrations with respect to $x$ and $y$ had been performed first no such difficulty would have been experienced, though it should be mentioned that, in view of the simplicity of the final result, the labour involved is quite considerable, even for a simple biconvex section.

It is evident that the method of proof given in the present note could easily be extended to cover the case of untapered swept wings. Again singular area distribution would occur when $\theta = \cos^{-1}(\beta^{-1}\tan\Lambda)$, where $\Lambda$ is the angle of sweep, but again they would not lead to serious difficulties.

## Appendix. Evaluation of two definite integrals

(i) $$\mathcal{I}(a) = \int_0^{\frac{1}{2}\pi} \sec^2\theta \log|1 - a\cos^2\theta|\, d\theta \quad (a > 0).$$

Write $\tan\theta = t$; then

$$\mathcal{I} = \int_0^\infty \log\left|\frac{t^2+1-a}{t^2+1}\right| dt.$$

(*a*) If $a < 1$, write $1 - a = c^2$; then

$$\begin{aligned}\mathcal{I} &= \int_0^\infty \log\left(\frac{t^2+c^2}{t^2+1}\right) dt \\ &= \left[t\log\left(\frac{t^2+c^2}{t^2+1}\right) + 2c\tan^{-1}\frac{t}{c} - 2\tan^{-1}t\right]_{t=0}^{\infty} \\ &= (c-1)\pi = -\pi\{1 - \sqrt{(1-a)}\}.\end{aligned}$$

(*b*) If $a > 1$, write $a - 1 = d^2$; then

$$\begin{aligned}\mathcal{I} &= \int_0^\infty \log\left|\frac{t^2-d^2}{t^2+1}\right| dt \\ &= \left[t\log\frac{|t^2-d^2|}{(t^2+1)} + d\log\left|\frac{t+d}{t-d}\right| - 2\tan^{-1}t\right]_{t=0}^{\infty} \\ &= -\pi.\end{aligned}$$

(ii) $$\begin{aligned}\mathcal{J}(a) &= \int_0^{\frac{1}{2}\pi} \sec\theta \log\left|\frac{1+a\cos\theta}{1-a\cos\theta}\right| d\theta \quad (a > 1) \\ &= \int_0^{\pi} \sec\theta \log|1 + a\cos\theta|\, d\theta.\end{aligned}$$

We shall first show that $\mathcal{J}(a)$ is independent of $a$, if $a \geqslant 1$. For

$$\frac{d}{da}\mathcal{J}(a) = \int_0^\pi \frac{d\theta}{(a\cos\theta - 1)},$$

where the Cauchy principal value must be taken. Writing $\tan\frac{1}{2}\theta = \tau$, we find that

$$\begin{aligned}\frac{d}{da}\mathcal{J}(a) &= \frac{2}{(1+a)}\int_0^\infty \frac{d\tau}{\alpha^2 - \tau^2}, \quad \text{where } \alpha = \sqrt{\left(\frac{a-1}{a+1}\right)}, \\ &= \frac{1}{\alpha(1+a)}\left[\log\left|\frac{\alpha+\tau}{\alpha-\tau}\right|\right]_{\tau=0}^{\infty} = 0.\end{aligned}$$

Hence
$$\mathcal{J}(a) = \mathcal{J}(1) = \int_0^{\frac{1}{2}\pi} \sec\theta \log\left(\frac{1-\cos\theta}{1+\cos\theta}\right) d\theta$$
$$= -4\int_0^1 \frac{\log\tau}{(1-\tau^2)}\,d\tau, \qquad \text{where } \tau = \tan\tfrac{1}{2}\theta,$$
$$= 2\int_0^1 \frac{1}{\tau}\log\left(\frac{1+\tau}{1-\tau}\right) d\tau$$
$$= 4\lim_{x\to 1}\int_0^x \left(1+\frac{\tau^2}{3}+\frac{\tau^4}{5}+\ldots\right) d\tau$$
$$= 4\lim_{x\to 1}\left(x+\frac{x^3}{3^2}+\frac{x^5}{5^2}+\ldots\right)$$
$$= \tfrac{1}{2}\pi^2.$$

The work described above was carried out in the Aerodynamics Division of the National Physical Laboratory and is published by permission of the Director of the Laboratory.

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